

On the Variant Boussinesq Equations

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We extend the generalized tanh method to the variant Boussinesq equations and obtain certain solitary-wave and new exact solutions.

The Boussinesq-typed equations belong to the most famous class of nonlinear evolution equations (see e.g., [1]). The set of the variant Boussinesq equations [2],

$$H_t + (H u)_x + u_{xxx} = 0, \quad (1)$$

$$u_t + H_x + u u_x = 0, \quad (2)$$

is one of the models for water waves. Lately, the set was found to possess travelling- or solitary-wave solutions with error [3] (to be explained later).

We now investigate (1) and (2) with an extended version of the generalized tanh method [4, 5] and symbolic computation, aiming at certain exact solutions beyond solitary waves. To begin with, we conjecture that those solutions be of the form

$$u(x, t) = \mathcal{A}(x, t) + \sum_{j=1}^N \mathcal{B}_j(t) \cdot \tanh^j[\Psi(t)x + \Theta(t)], \quad (3)$$

$$H(x, t) = \mathcal{C}(x, t) + \sum_{j=1}^M \mathcal{D}_j(t) \cdot \tanh^j[\Psi(t)x + \Theta(t)], \quad (4)$$

where M and N are integers, and $\mathcal{A}(x, t)$, $\mathcal{B}_j(t)$'s, $\mathcal{C}(x, t)$, $\mathcal{D}_j(t)$'s, $\Psi(t)$ and $\Theta(t)$ are differentiable functions. The x -linear proposal is based on the consideration that the physical fields $H(x, t)$ and $u(x, t)$ have only a first derivative with respect to t but higher-order derivatives with respect to x . Our extension hereby is to assume that $\mathcal{A}(x, t)$ and $\mathcal{C}(x, t)$ are functions not only of t but also of x .

On balancing the highest-order contributions from the linear terms with the highest order contributions from the nonlinear terms, we get $M = 2$ and $N = 1$, so that

$$u(x, t) = \mathcal{A}(x, t) + \mathcal{B}_1(t) \cdot \tanh[\Psi(t)x + \Theta(t)], \quad (5)$$

$$H(x, t) = \mathcal{C}(x, t) + \mathcal{D}_1(t) \cdot \tanh[\Psi(t)x + \Theta(t)] + \mathcal{D}_2(t) \cdot \tanh^2[\Psi(t)x + \Theta(t)], \quad (6)$$

where $\mathcal{B}_1(t) \neq 0$, $\mathcal{D}_2(t) \neq 0$ and $\Psi(t) \neq 0$.

Symbolically computing (1) and (2) with (5) and (6) yields

$$\begin{aligned} & \mathcal{B}_1 \mathcal{C} \Psi \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{A} \mathcal{D}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) - 2 \mathcal{B}_1 \Psi^3 \operatorname{sech}^4(\Psi x + \Theta) + \mathcal{C}_t + \mathcal{C} \mathcal{A}_x \\ & + 2 \mathcal{B}_1 \mathcal{D}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + 2 \mathcal{A} \mathcal{D}_2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) \\ & + 3 \mathcal{B}_1 \mathcal{D}_2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh^2(\Psi x + \Theta) + 4 \mathcal{B}_1 \Psi^3 \operatorname{sech}^2(\Psi x + \Theta) \tanh^2(\Psi x + \Theta) \\ & + \mathcal{D}_{1,t} \tanh(\Psi x + \Theta) + \mathcal{D}_{2,t} \tanh^2(\Psi x + \Theta) + x \mathcal{D}_1 \Psi_t \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{A} \mathcal{C}_x + \mathcal{A}_{xxx} \\ & + 2 x \mathcal{D}_2 \Psi_t \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + 2 \mathcal{D}_2 \Theta_t \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) \\ & + \mathcal{D}_1 \Theta_t \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{D}_1 \mathcal{A}_x \tanh(\Psi x + \Theta) + \mathcal{D}_2 \mathcal{A}_x \tanh^2(\Psi x + \Theta) \\ & + \mathcal{B}_1 \mathcal{C}_x \tanh(\Psi x + \Theta) = 0, \end{aligned} \quad (7)$$

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$$\begin{aligned} & \mathcal{A} \mathcal{B}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{D}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{B}_1^2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + \mathcal{A} \mathcal{A}_x \\ & + 2 \mathcal{D}_2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + \mathcal{B}_{1,t} \tanh(\Psi x + \Theta) + x \mathcal{B}_1 \Psi_t \operatorname{sech}^2(\Psi x + \Theta) \\ & + \mathcal{B}_1 \Theta_t \operatorname{sech}^2(\Psi x + \Theta) + \mathcal{A}_t + \mathcal{B}_1 \mathcal{A}_x \tanh(\Psi x + \Theta) + \mathcal{C}_x = 0. \end{aligned} \quad (8)$$

When we equate to zero the coefficients of like powers of $\tanh(\Psi x + \Theta)$, after some algebraic manipulations (7) and (8) turn out to be

$$\tanh^4(\Psi x + \Theta), \tanh^3(\Psi x + \Theta): \quad \mathcal{D}_2(t) = -2\Psi^2(t), \quad \mathcal{B}_1(t) = \pm 2\Psi(t), \quad (9)$$

$$\tanh^3(\Psi x + \Theta), \tanh^2(\Psi x + \Theta): \quad \mathcal{A}(x, t) = \frac{\pm \mathcal{D}_1(t) - \Theta_t(t) - \Psi_t(t)x}{\Psi(t)}, \quad \mathcal{D}_1(t) = 0, \quad (10)$$

$$\tanh^2(\Psi x + \Theta), \dots: \quad \mathcal{C} = 2\Psi^2(t) \mp \frac{\Psi_t(t)}{\Psi(t)} = \mathcal{C}(t) \quad \text{only}, \quad (11)$$

$$\tanh(\Psi x + \Theta), \tanh(\Psi x + \Theta): \quad \text{satisfied automatically}. \quad (12)$$

Then, equating to zero the coefficients of like powers of x with integrations gives rise to

$$x^0, x: \quad \Psi(t) = \frac{1}{\alpha t + \beta} \quad (\beta \neq 0), \quad (13)$$

$$-, x^0: \quad \Theta(t) = \begin{cases} \frac{\gamma}{\alpha t + \beta} + \delta & \text{if } \alpha \neq 0, \\ \zeta t + \eta & \text{otherwise,} \end{cases} \quad (14)$$

where $\alpha, \beta, \gamma, \delta, \zeta$ and η are integration constants.

The last step is to substitute everything back into (5) and (6), so that we obtain a couple of families of analytical solutions for (1) and (2), as follows:

Family I: Solitary-wave solutions with $\alpha = 0$

$$u^I(x, t) = \pm \frac{2}{\beta} \tanh\left(\frac{x}{\beta} + \zeta t + \eta\right) - \beta \zeta, \quad (15)$$

$$H^I(x, t) = \frac{2}{\beta^2} \operatorname{sech}^2\left(\frac{x}{\beta} + \zeta t + \eta\right). \quad (16)$$

It is easy to show that the solutions found in [3] are special cases of (15) and (16) with the choice of a “+” sign in (15). It is also noted that (25) in [3] is incorrect.

Family II: New exact solutions with $\alpha \neq 0$ and $|x| \neq \infty$ (17)

$$u^{II}(x, t) = \frac{1}{\alpha t + \beta} \left[\pm 2 \cdot \tanh\left(\frac{x + \gamma}{\alpha t + \beta} + \delta\right) + \alpha(x + \gamma) \right],$$

$$H^{II}(x, t) = \frac{2}{(\alpha t + \beta)^2} \operatorname{sech}^2\left(\frac{x + \gamma}{\alpha t + \beta} + \delta\right) \pm \frac{\alpha}{\alpha t + \beta}. \quad (18)$$

Those families provide information on the motion of water waves.

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