On the Variant Boussinesq Equations

Yi-Tian Gao and Bo Tian

Department of Applied Mathematics and Physics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China

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We extend the generalized tanh method to the variant Boussinesq equations and obtain certain solitary-wave and new exact solutions.

The Boussinesq-typed equations belong to the most famous class of nonlinear evolution equations (see e.g., [1]). The set of the variant Boussinesq equations [2],

$$H_t + (H u)_x + u_{xxx} = 0\,, \tag{1}$$

$$u_t + H_x + u u_x = 0\,, \tag{2}$$

is one of the models for water waves. Lately, the set was found to possess travelling- or solitary-wave solutions with error [3] (to be explained later).

We now investigate (1) and (2) with an extended version of the generalized tanh method [4, 5] and symbolic computation, aiming at certain exact solutions beyond solitary waves. To begin with, we conjecture that those solutions be of the form

$$u(x, t) = \mathscr{A}(x, t) + \sum_{j=1}^{N} \mathscr{B}_j(t) \cdot \tanh^j[\Psi(t)\,x + \Theta(t)]\,, \tag{3}$$

$$H(x, t) = \mathscr{C}(x, t) + \sum_{j=1}^{M} \mathscr{D}_j(t) \cdot \tanh^j[\Psi(t)\,x + \Theta(t)]\,, \tag{4}$$

where M and N are integers, and $\mathscr{A}(x, t)$, $\mathscr{B}_j(t)$'s, $\mathscr{C}(x, t)$, $\mathscr{D}_j(t)$'s, $\Psi(t)$ and $\Theta(t)$ are differentiable functions. The x-linear proposal is based on the consideration that the physical fields $H(x, t)$ and $u(x, t)$ have only a first derivative with respect to t but higher-order derivatives with respect to x. Our extension hereby is to assume that $\mathscr{A}(x, t)$ and $\mathscr{C}(x, t)$ are functions not only of t but also of x.

On balancing the highest-order contributions from the linear terms with the highest order contributions from the nonlinear terms, we get $M = 2$ and $N = 1$, so that

$$u(x, t) = \mathscr{A}(x, t) + \mathscr{B}_1(t) \cdot \tanh[\Psi(t)\,x + \Theta(t)]\,, \tag{5}$$

$$H(x, t) = \mathscr{C}(x, t) + \mathscr{D}_1(t) \cdot \tanh[\Psi(t)\,x + \Theta(t)] + \mathscr{D}_2(t) \cdot \tanh^2[\Psi(t)\,x + \Theta(t)]\,, \tag{6}$$

where $\mathscr{B}_1(t) \neq 0$, $\mathscr{D}_2(t) \neq 0$ and $\Psi(t) \neq 0$.

Symbolically computing (1) and (2) with (5) and (6) yields

$$\begin{aligned}
&\mathscr{B}_1 \mathscr{C} \Psi \operatorname{sech}^2(\Psi x + \Theta) + \mathscr{A} \mathscr{D}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) - 2 \mathscr{B}_1 \Psi^3 \operatorname{sech}^4(\Psi x + \Theta) + \mathscr{C}_t + \mathscr{C} \mathscr{A}_x \\
&\quad + 2 \mathscr{B}_1 \mathscr{D}_1 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + 2 \mathscr{A} \mathscr{D}_2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) \\
&\quad + 3 \mathscr{B}_1 \mathscr{D}_2 \Psi \operatorname{sech}^2(\Psi x + \Theta) \tanh^2(\Psi x + \Theta) + 4 \mathscr{B}_1 \Psi^3 \operatorname{sech}^2(\Psi x + \Theta) \tanh^2(\Psi x + \Theta) \\
&\quad + \mathscr{D}_{1,t} \tanh(\Psi x + \Theta) + \mathscr{D}_{2,t} \tanh^2(\Psi x + \Theta) + x \mathscr{D}_1 \Psi_t \operatorname{sech}^2(\Psi x + \Theta) + \mathscr{A} \mathscr{C}_x + \mathscr{A}_{xxx} \\
&\quad + 2 x \mathscr{D}_2 \Psi_t \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) + 2 \mathscr{D}_2 \Theta_t \operatorname{sech}^2(\Psi x + \Theta) \tanh(\Psi x + \Theta) \\
&\quad + \mathscr{D}_1 \Theta_t \operatorname{sech}^2(\Psi x + \Theta) + \mathscr{D}_1 \mathscr{A}_x \tanh(\Psi x + \Theta) + \mathscr{D}_2 \mathscr{A}_x \tanh^2(\Psi x + \Theta) \\
&\quad + \mathscr{B}_1 \mathscr{C}_x \tanh(\Psi x + \Theta) = 0\,,
\end{aligned} \tag{7}$$

Reprint requests and correspondence to Prof. Bo Tian.

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$$\mathcal{A}\,\mathcal{B}_1\,\Psi\,\mathrm{sech}^2(\Psi x+\Theta)+\mathcal{D}_1\,\Psi\,\mathrm{sech}^2(\Psi x+\Theta)+\mathcal{B}_1^2\,\Psi\,\mathrm{sech}^2(\Psi x+\Theta)\tanh(\Psi x+\Theta)+\mathcal{A}\,\mathcal{A}_x$$
$$+2\,\mathcal{D}_2\,\Psi\,\mathrm{sech}^2(\Psi x+\Theta)\tanh(\Psi x+\Theta)+\mathcal{B}_{1,t}\tanh(\Psi x+\Theta)+x\,\mathcal{B}_1\,\Psi_t\,\mathrm{sech}^2(\Psi x+\Theta)$$
$$+\mathcal{B}_1\,\Theta_t\,\mathrm{sech}^2(\Psi x+\Theta)+\mathcal{A}_t+\mathcal{B}_1\,\mathcal{A}_x\tanh(\Psi x+\Theta)+\mathcal{C}_x=0\,. \tag{8}$$

When we equate to zero the coefficients of like powers of $\tanh(\Psi x+\Theta)$, after some algebraic manipulations (7) and (8) turn out to be

$$\tanh^4(\Psi x+\Theta),\ \tanh^3(\Psi x+\Theta):\qquad \mathcal{D}_2(t)=-2\,\Psi^2(t)\,,\qquad \mathcal{B}_1(t)=\pm 2\,\Psi(t)\,, \tag{9}$$

$$\tanh^3(\Psi x+\Theta),\ \tanh^2(\Psi x+\Theta):\qquad \mathcal{A}(x,t)=\frac{\pm\,\mathcal{D}_1(t)-\Theta_t(t)-\Psi_t(t)\,x}{\Psi(t)}\,,\qquad \mathcal{D}_1(t)=0\,, \tag{10}$$

$$\tanh^2(\Psi x+\Theta),\ ---------:\qquad \mathcal{C}=2\,\Psi^2(t)\mp\frac{\Psi_t(t)}{\Psi(t)}=\mathcal{C}(t)\quad\text{only}\,, \tag{11}$$

$$\tanh(\Psi x+\Theta),\ \tanh(\Psi x+\Theta):\qquad \text{satisfied automaticlly}\,. \tag{12}$$

Then, equating to zero the coefficients of like powers of x with integrations gives rise to

$$x^0,\,x:\quad \Psi(t)=\frac{1}{\alpha t+\beta}\qquad(\beta\neq 0)\,, \tag{13}$$

$$-,\,x^0:\quad \Theta(t)=\begin{cases}\dfrac{\gamma}{\alpha t+\beta}+\delta & \text{if}\quad \alpha\neq 0\,,\\ \zeta t+\eta & \text{otherwise}\,,\end{cases} \tag{14}$$

where α, β, γ, δ, ζ and η are integration constants.

The last step is to substitute everything back into (5) and (6), so that we obtain a couple of families of analytical solutions for (1) and (2), as follows:

Family I: Solitary-wave solutions with $\alpha=0$

$$u^I(x,t)=\pm\frac{2}{\beta}\tanh\left(\frac{x}{\beta}+\zeta t+\eta\right)-\beta\zeta\,, \tag{15}$$

$$H^I(x,t)=\frac{2}{\beta^2}\,\mathrm{sech}^2\left(\frac{x}{\beta}+\zeta t+\eta\right). \tag{16}$$

It is easy to show that the solutions found in [3] are special cases of (15) and (16) with the choice of a "+" sign in (15). It is also noted that (25) in [3] is incorrect.

Family II: New exact solutions with $\alpha\neq 0$ and $|x|\neq\infty$

$$u^{II}(x,t)=\frac{1}{\alpha t+\beta}\left[\pm 2\cdot\tanh\left(\frac{x+\gamma}{\alpha t+\beta}+\delta\right)+\alpha(x+\gamma)\right], \tag{17}$$

$$H^{II}(x,t)=\frac{2}{(\alpha t+\beta)^2}\,\mathrm{sech}^2\left(\frac{x+\gamma}{\alpha t+\beta}+\delta\right)\pm\frac{\alpha}{\alpha t+\beta}\,. \tag{18}$$

Those families provide information on the motion of water waves.

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